Stochastic Processes II
Spring 2015
FINAL EXAM
May 18, 2015

Do all problems from 1 to 8 and anyone from 9 to 10.
Each part has 5 points. Maximum possible points is 75.

1. (a) For a standard one-dimensional Brownian motion $\{B_t\}_{t\geq 0}$, show that $B_t^2 - t$ is a martingale.
   (b) Using the facts that B_t is a martingale and it has strong Markov property compute the probability $P(T_2 < T_{-1} < T_4)$, where $T_a := \inf\{t \geq 0 : B_t = a\}$.

2. (a) Consider a barbershop with two barbers and two waiting chairs. Customers arrive at a rate of 5 per hour. Customers arriving to a fully occupied shop leave without being served. Find the stationary distribution for the number of customers in the shop, assuming that the service rate for each barber is 2 customers per hour.

   (b) Now consider a barbershop with one barber who can cut hair at rate 4 and three waiting chairs. Customers arrive at a rate of 5 per hour. Compute the increase in the number of customers served per hour.

3. Consider a M/M/1 queue with arrival rate λ, service rate μ. Assume that $\lambda < \mu$. Compute the long run average time spent by a customer in the system. Also compute the long run average time spent by a customer standing in the queue. (Remember that the stationary distribution for the number of customers in the system is Geometric(λ/μ))

4. Each time the frozen yogurt machine at the mall breaks down, it is replaced by a new one of the same type. What is the limiting age distribution for the machine in use if the lifetime of a machine has a gamma(2,λ) distribution, i.e., the sum of two exponentials with mean $1/\lambda$.

5. People arrive at a college admissions office at rate 1 per minute. When k people have arrive a tour starts. Student tour guides are paid \$20 for each tour they conduct. The college estimates that it loses 10 cents in good will for each minute a person waits. What is the optimal tour group size?

6. Hockey teams 1 and 2 score goals at times of Poisson processes with rates λ_1 and λ_2. Suppose that $N_1(0) = 3$ and $N_2(0) = 1$. Find an expression for the probability that $N_1(t)$ will reach 5 before $N_2(t)$ does?

7. Rock concert tickets are sold at a ticket counter. Females and males arrive at times of independent Poisson processes with rates 30 and 20 customers per hour. (a) If exactly 2 customers arrived in the first five minutes, what is the probability both arrived in the first three minutes. (b) Suppose that customers regardless of sex buy 1 ticket with probability 1/2, two tickets with probability 2/5, and three tickets with probability 1/10. Let N_i be the number of customers that buy i tickets in the first hour. Find the joint distribution of (N_1, N_2, N_3). (c) If each ticket costs \$20, find the mean and variance of the amount of business done with females during the afternoon session (1 pm to 4 pm).

8. Gambler's ruin problem. Let $S_n = x + X_1 + \cdots + X_n$ be the fortune of a gambler after n steps with initial fortune x, where $X_1, X_2, \ldots$ are independent with $P(X_i = 1) = p < 1/2$ and $P(X_i = -1) = 1 - p$. Show that $[(1-p)/p]^{S_n}$ is a martingale. Using this compute the probability that the gambler reaches \$100 before ruin starting with \$50.

9. Given the transition matrix P of a finite irreducible discrete time Markov chain $\{X_t\}$ on state space $S = \{0, 1, 2, 3, 4\}$, how will you obtain $E_x T_y$ for $x, y \in S$, where $T_y := \min\{k \geq 1 : X_k = y\}$.

10. A submarine has three navigational devices but can remain at sea if at least two are working. Suppose that the failure times are exponential with means 1 year, 1.5 years, and 3 years. Formulate a Markov chain with states 0= all parts working, 1,2,3 = one part failed, and 4 = two failures. Compute $E_0 T_4$ to determine the average length of time the boat can remain at sea.